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Howard and D'Antonio (1984) developed the hedge ratio and the measure of hedging effectiveness of futures contracts in the framework of what they called the modern portfolio theory. This note shows that the H-D analysis is misleading and not consistent with the portfolio theory. For the comparison purpose, an alternative and simpler hedge ratio and measure of hedging effectiveness of futures is developed which is consistent with the portfolio theory.

# Hedging in the Portfolio Theory Framework: A Note

Hun Y. Park

## I. Introduction

The key to any hedging strategy using futures contracts is a knowledge of the hedge ratio, i.e., the number of futures per spot position. The most common method to estimate the hedge ratio using futures contracts is the regression approach relating changes in cash prices to changes in futures prices. Inherent in the regression is the assumption that the optimal combination of cash position with futures is the one whose variance is minimized. This method was originally developed by Johnson (1960) and Stein (1961) and has been used by a number of subsequent studies (e.g., Ederington (1979)).

Recently, Howard and D'Antonio (1984) criticized the minimum-variance approach on the ground that it lacks the consideration of the risk-return tradeoff, and suggested an alternative method of determining the hedge ratio and the hedging effectiveness in the framework of what they called the modern portfolio theory. To the author's best knowledge, the paper is the first theoretical attempt to take into account the risk-return tradeoff in determining the hedge ratio and the hedging effectiveness of futures contracts. Consideration of the risk-return tradeoff of a hedged portfolio relative to that of an unhedged portfolio is important since hedging should be viewed as an activity that sacrifices the expected return in exchange for the lower risk. However, the hedge ratio and hedging effectiveness in Howard and D'Antonio (1984) is misleading and not consistent with the modern portfolio theory.

The purpose of this note is to show the pitfalls of the Howard and D'Antonio analysis and to provide an alternative and simpler hedge ratio and measure of hedging effectiveness which is consistent with the portfolio theory.

## II. Howard and D'Antonio Model

The H-D hedge ratio was derived by maximizing $\theta$

$$\theta = \frac{\overline{R}_p - i}{\sigma_p} \tag{1}$$

where

$\overline{R}_p$ = the expected return for the spot and futures portfolio

$i$ = the risk-free rate of return

$\sigma_p$ = the standard deviation of the return for the spot and futures portfolio

The first-order condition for maximization of $\theta$ in (1) gives the H-D hedge ratio (i.e., the number of units of the futures per the spot unit) as

$$b^* = \frac{\lambda - \rho}{\gamma\pi(1-\lambda\rho)}\ , \tag{2}$$

where $\lambda = \dfrac{\overline{r}_f}{\sigma_f} / \dfrac{\overline{r}_s - i}{\sigma_s}$

$\pi = \sigma_f/\sigma_s$

$\alpha = \overline{r}_f/(\overline{r}_s - i)$

$\gamma = F/S$

$S, F$ = the current price per unit for the spot and futures, respectively

$\bar{r}_s, \bar{r}_f$ = the expected one-period returns for the spot and futures, respectively

$\sigma_s, \sigma_f$ = the standard deviation of one-period returns for the spot and futures, respectively

$\rho$ = the correlation between the returns of the spot and futures.

Using (2), they derived the measure of hedging effectiveness (HE), defined as the ratio of the slope of the line connecting the risk free asset's return, i and the portfolio to that of the line connecting i and the spot position in the return-standard deviation space as

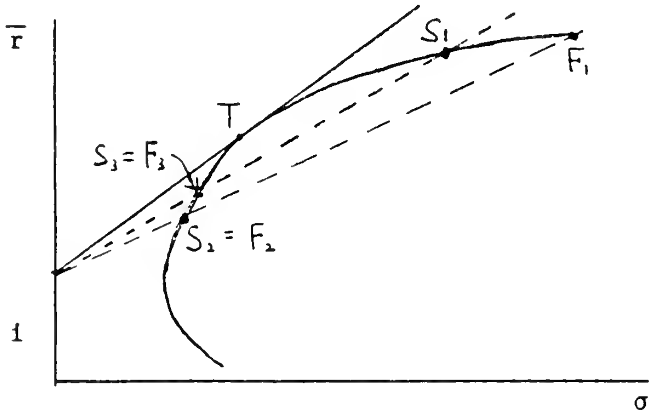
$$HE = \frac{(\bar{r}_p - i)}{\sigma_p} / \frac{(\bar{r}_s - i)}{\sigma_s} = \sqrt{\frac{1 - 2\lambda\rho + \lambda^2}{1 - \rho^2}} \tag{3}$$

Note that this measure is no more than a Sharpe's measure equivalent of portfolio performance. The fundamental problem in the H-D approach is that they ignore the fact that futures contracts are <u>not</u> different in principle from other risky assets (see Dusak (1973)). Futures contracts as well as other risky assets must be included in the market portfolio in the Markowitz framework. Note that H-D attempted to compute the number of futures contracts in such a way that the combination of spot and futures be on the efficient frontier.[1] The underlying assumption is that neither spot nor futures per se may be efficient in the mean-variance space. Thus, if both futures and spot positions are included in the market portfolio and thus efficient, Howard and D'Antonio argue that "futures do not provide a benefit" for hedging,

which is groundless in the modern portfolio theory. If futures contracts are included in the market portfolio just like other risky assets, the use of futures for hedging spot positions should be determined from consideration of movements of spot and futures prices relative to the market portfolio.

The problems in the H-D analysis are more clear when we examine the H-D hedge ratio in (2). As argued in H-D, the hedge ratio depends on $\lambda$, the risk-return relative, and its relation with $\rho$, the correlation between futures and spot prices for the given $\pi$, the ratio of $\sigma_f$ to $\sigma_s$, and $\gamma$, the ratio of the futures price to the spot price: If $\lambda$ is less (greater) than $\rho$, then a short (long) position is optimal. If $\lambda$ equals $\rho$, the optimal hedge ratio, $b^*$, is zero so that there is no benefit to holding futures for hedging purpose.

However, consider the positions of spot and futures in the following figure.




First, the H-D analysis is based on the assumption that $\lambda < 1$, i.e., the spot position has always a superior return-risk property relative to futures contracts.[2] This would be the case in the above figure where the spot position is $S_1$ and the futures is either $F_1$ or $F_2$.

However, there is no economic reason whatsoever why the spot position should have better return-risk tradeoff than futures. For example, consider the spot position $S_2$ and the futures position $F_3$, which is the case where $\lambda > 1$. We can imagine that the tangent portfolio T can be constructed, depending on the correlation between $S_2$ and $F_3$. Furthermore, in the H-D analysis, the only case where a short position in futures is optimal is when $\lambda < 1$ and $\lambda < \rho$. But, consider the positions $S_1$ and $F_2$ for the spot and futures, respectively, and assume $\lambda = .5$ and $\rho = .8$. It is obvious that we should not take a short position in $F_2$ to get the tangent portfolio T. If the spot position is $S_1$, the futures should be $F_1$ in the above figure for the short position in futures to be taken. Thus, we need another restriction that the futures position should be somewhere in the right hand side of $S_1$, if we follow the H-D. Also, even in the above case of $\lambda < 1$ (i.e., $S_1$ and $F_1$ for the spot and futures positions, respectively), H-D argue that if $\lambda > \rho$, then investors should take long positions in futures for hedging purpose. What if $\lambda$ equals .5 and $\rho$ equals any number less than .5? Obviously, we can imagine investors should take short positions in futures ($F_1$) to reduce the risk of the spot position ($S_1$). For the same reason, the argument that futures is useless for hedging purpose if $\lambda = \rho$ is not quite correct.

What happens to the hedge ratio if the spot and futures positions are on the same line starting from i (e.g., $S_1$ and $F_3$ or $S_2$ and $F_1$)? Obviously, $\lambda = 1$ and thus the hedge $b^*$ in (2) narrows down to $1/\gamma\pi$, which is not a function of $\rho$. This argument leads to the unreasonable conclusion that the correlations between spot and futures do <u>not</u> matter. Note also that if $\rho = 1$, $b^* = -1/\gamma\pi$. However, it is well

known that if $\rho = 1$, we cannot obtain the portfolio T through diversification, unlike the H-D claim. Furthermore, if $\lambda = \rho = 1$, the hedge ratio ($b^*$) is not defined. H-D argued that this case is of little practical importance.[3] However, it is quite conceivable when the underlying security is identical to the hedged security and there is no basis risk.

Also, note that the second-order condition for the $b^*$ derivation is that $\lambda\rho < 1$. However, we may observe many cases where $\rho$ is close to 1 and $\lambda$ is much higher than 1, so that $\lambda\rho$ is greater than or equal to 1.

## III. Hedge Ratio and Hedging Effectiveness in the Portfolio Theory

Acknowledging that futures contracts are not different in principle from any other risky assets, the following simple and portfolio theory consistent hedge ratio can be developed in the CAPM context.

By definition,

$$R_p = \frac{\Delta S + D + N\Delta F}{S}, \tag{4}$$

where $\Delta S$ = the change in spot prices.

$D$ = dividends on the underlying stock

$N$ = the hedge ratio

$\Delta F$ = the change in futures prices

Then

$$R_p = \frac{\Delta S + D}{S} + N \cdot \frac{F}{S} \cdot \frac{\Delta F}{F} \tag{4'}$$

$$= r_s + N \frac{F}{S} \cdot r_f$$

The beta of the portfolio p containing S and F can thus be written as

$$\beta_p = \beta_s + N(F/S)\beta_f \tag{5}$$

Therefore, the optimal hedge ratio, N, equals

$$N = \frac{(\beta_p - \beta_s)}{\beta_f} \cdot \frac{S}{F} \tag{6}$$

Note that the optimal hedge ratio in (6) depends on the desired level of beta of the portfolio ($\beta_p$) as well as the betas of S and F. This hedge ratio is more plausible than the H-D not only because it is consistent with the modern portfolio theory but also because the basis risk relative to the market portfolio is explicitly incorporated (i.e., the relation between S(F) and M are reflected in $\beta_S(\beta_F)$ and thus the relative basis risk, the relation between S and F.)

If the market is expected to be bullish, the portfolio managers may want to increase their portfolio's beta position in such a way that $\beta_p > \beta_s$. If the market is expected to be bearish, the portfolio managers can decrease the exposure to the market risk by adjusting $\beta_p$ to be less than $\beta_s$. On the other hand, if the manager wants to completely eliminate the systematic risk of the portfolio, $N = -\frac{\beta_s}{\beta_f} \cdot \frac{S}{F}$, which yields the so-called zero-beta portfolio.

From the above discussion, a measure of hedging effectiveness can be developed as follows:

$$HE^* = \left(\frac{\bar{R}_p - i}{\beta_p}\right) / \left(\frac{\bar{R}_s - i}{\beta_s}\right) \tag{7}$$

If the H-D measure in (3) is similar to "Sharpe's measure" of portfolio performance, the measure in (7) is similar to the "Treynor measure" of portfolio performance. Note, however, that Sharpe's measure can be used only for efficient portfolios, which is contradictory to the underlying assumptions of H-D.

Alternatively, since we can construct the zero beta portfolios using futures and spot, or spot securities only, the hedging effectiveness can be measured using:

$$HE^* = r^*_{OA} / r^*_{OB} , \tag{8}$$

where $r^*_{OA}$ = the return on the zero-beta portfolio using futures and spot positions as dictated by (6)

$r^*_{OB}$ = the return on the zero-beta portfolio constructed by using only spot securities.

## IV. Concluding Remarks

Howard and D'Antonio (1984) developed the hedge ratio and the measure of hedging effectiveness of futures contracts in the framework of what they called the modern portfolio theory. The fundamental problem in the H-D analysis is that they ignore the fact that futures contracts are not different in principle from other risky assets, so that futures as well as spot securities must be included in the market portfolio in the Markowitz framework. This note shows that the H-D

analysis is misleading and not consistent with the modern port-folio theory. In addition, for the comparison purpose, an alternative simpler hedge ratio and measure of hedging effectiveness of futures is provided which is consistent with the portfolio theory.

## Footnotes

[1]See Graphs A, B and C in Figure 1, pp. 103 of Howard and D'Antonio (1984).

[2]Note that $\lambda$ and $\alpha$ were derived on the assumption that futures contracts require zero initial margins. If futures contracts were treated in the same way as underlying spot securities, $\lambda$ and $\alpha$ would be

$\frac{\bar{r}_f - i}{\sigma_f} / \frac{\bar{r}_s - i}{\sigma_s}$ and $(\bar{r}_f - i)/(\bar{r}_s - i)$, respectively. See footnotes 6 and 8 in Howard and D'Antonio (1984). For simplicity, this paper uses

$\frac{\bar{r}_f - i}{\sigma_f} / \frac{\bar{r}_s - i}{\sigma_s}$ and $(\bar{r}_f - i)/(\bar{r}_s - i)$ for $\lambda$ and $\alpha$, respectively. However, the analysis would be intact.

[3]See footnote 10, pp. 107 in Howard and D'Antonio (1984).

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